UNIVERSAL VALUED FIELDS AND LIFTING POINTS IN LOCAL TROPICAL VARIETIES

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ABSTRACT. Let k be a field with a real valuation ν and R a k-algebra. We show that there exist a k-algebra K and a valuation μ on K extending ν such that any real valuation of R is induced by μ via some homomorphism from R to K. Now let ν be trivial and R a complete local Noetherian ring with the residue field k. Let K be the ring $\bar{k}[[t^{\mathbb{R}}]]$ of Hahn series with its natural valuation μ and coefficients in $\bar{k}$. We prove the following weak universality property: for any local valuation v and a finite set of elements $x_1, \ldots, x_n$ of R there exists a homomorphism $f\colon R \to K$ such that $v(x_i) = \mu(f(x_i))$, $i = 1, \ldots, n$. This implies that if $R = k[[x_1, \ldots, x_n]]/I$ for an ideal I, then every point of the local tropicalization of I lifts to a K-point of R.

1. Introduction

Let k be a field and R a commutative k-algebra with unity. A *real* (or *rank one*) *valuation* on R is a function $v\colon R \to \mathbb{R} \cup \{+\infty\}$ such that $v(1) = 0$, $v(0) = +\infty$, $v(xy) = v(x) + v(y)$ and $v(x + y) \geq \min\{v(x), v(y)\}$ for all x, $y \in R$. In this paper we consider only real ring valuations. Assume that k is endowed with some valuation ν and K is some k-algebra with a valuation μ extending ν. Then every homomorphism $f\colon R \to K$ of k-algebras induces a valuation $\mu \circ f$ of R extending ν. The first question we address in this paper is the following: given R and ν, does there exist a universal valued field K such that *any* valuation of R that extends ν is obtained via some homomorphism $f\colon R \to K$ as described above? We give an affirmative answer to this question in Theorem 2.7 of Section 2. Note that the universal field we are looking for is different from the maximal immediate extension of k ([Krull, p. 191]) since we do not insist that v must have the same residue field as ν.

The second problem that we study in this work is lifting points in local tropical varieties. If R is a local ring with the maximal ideal $\mathfrak{m}$, we call a valuation v of R *local*, if v is nonnegative on R and positive on $\mathfrak{m}$. Note that if R contains a field k, any local valuation of R must be trivial on k, i.e., $v(x) = 0$ for all $x \in k$, $x \neq 0$. Now let $R = k[[x_1, \ldots, x_n]]$ be the ring of formal power series in variables $x_1, \ldots, x_n$, and $I \subset R$ an ideal. The *local tropicalization* $\mathrm{Trop}_{>0}(I)$ of I is the set

$$\mathrm{Trop}_{>0}(I) = \{(v(x_1), \ldots, v(x_n) \,|\, v \text{ is local, } v|_I = +\infty\} \subseteq \overline{\mathbb{R}}^n,$$

where v runs over all local valuations of R that take value $+\infty$ on the ideal I, and $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$. This is a particular case of [PPS, Definition 6.6]; see also Section 4. Let us introduce two special fields. Denote by $\bar{k}$ some algebraic closure of k. The field $K = \bar{k}((t^{Q})) = \bigcup_{N \geq 1} \bar{k}((t^{1/N}))$ of *Puiseux series* with coefficients in $\bar{k}$ is the

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set of all formal sums

$$\sum_{m\in\mathbb{Q}} a_m t^m,$$

where $a_m \in \bar{k}$, the set $\{m \mid a_m \neq 0\} \subset \mathbb{Q}$ is bounded from below, and all its elements have bounded denominators. The field $K' = \bar{k}((t^{\mathbb{R}}))$ of *Hahn series* is the set of all formal sums

$$a(t) = \sum_{m\in\mathbb{R}} a_m t^m,$$

where the set $\{m \mid a_m \neq 0\}$ is well-ordered, that is, any its subset has the least element. Addition and multiplication in K and K' are defined in a natural way. These fields carry also a natural valuation μ determined by $\mu(t) = 1$. The *ring of Puiseux* (respectively *Hahn*) *series* $\mathcal{O} = \bar{k}[[t^{\mathbb{Q}}]]$ (resp., $\mathcal{O}' = \bar{k}[[t^{\mathbb{R}}]]$) is the valuation ring of μ, i.e., the subring of K (resp. K') where μ is nonnegative. In analogy with the theory of usual non-local tropicalization (see, e.g., [JMM]), we say that a point $w = (w_1, \dots, w_n) \in \mathrm{Trop}_{>0}(I) \cap \overline{\mathbb{Q}}^n$, where $\overline{\mathbb{Q}} = \mathbb{Q} \cup \{+\infty\}$, *lifts to* $\mathcal{O}$, if there exists a homomorphism $f\colon R \to K$ such that $w_i = \mu(f(x_i))$ for all $i = 1, \dots, n$. Lifting of a point $w \in \mathrm{Trop}_{>0}(I)$ to $\mathcal{O}'$ can be defined similarly. In the non-local theory of tropicalization, it is known that if characteristic of k is 0, then each rational point of the tropicalization of a variety admits a lifting to a K-point of the variety. Several proofs and generalizations are given in [Draisma], [JMM], [Katz], [Payne]. For $k = \mathbb{C}$ and local tropical varieties, a similar result was announced by N. Touda in [Touda], however, a complete proof did not appear.

The possibility to lift points of local tropical varieties to $\mathcal{O}$ or $\mathcal{O}'$ means that these rings play a role of a kind of universal domains with respect to valuations on local k-algebras R, where the field k is trivially valued. This is not the strong universality introduced in the first paragraph of this Introduction. Indeed, if we set $R = k[[x, y]]$ and v to be a monomial valuation defined by $v(x) = v(y) = 1$, then it is easy to see that v is not induced by any homomorphism from R to $\mathcal{O}$ or $\mathcal{O}'$. However, it is easy to define a homomorphism $f\colon R \to \mathcal{O}$, say, by sending x and y to t, such that $\mu(f(x)) = v(x)$, $\mu(f(y)) = v(y)$. Precisely, we have the following property, which we call *weak universality* of the rings of Puiseux and Hahn series. Assume that R is a complete local Noetherian ring, k is a field isomorphic to the residue field of R and contained in R, and $x_1, \dots, x_n \in R$ a finite collection of elements. Then for each local valuation v of R there exists a homomorphism $f\colon R \to \mathcal{O}'$ such that $v(x_i) = \mu(f(x_i))$ for all $i = 1, \dots, n$. If, moreover, k has characteristic 0 and v takes only rational values, then there exists a homomorphism $f\colon R \to \mathcal{O}$ with the same property. Similarly, lifting points in the non-local tropical varieties can be expressed as the weak universality of the *fields* of Puiseux and Hahn series with respect to valuations on finitely generated k-algebras R. We prove these properties in Section 3 and then apply them to the local tropicalization in Section 4.

Our proof of the weak universality of the Puiseux and Hahn series rings is closed to the proof of the lifting points property of tropical varieties given in [JMM]. We also use a descent by dimension, but we descent not to dimension 0 but to dimension r equal to the rational rank of the value group of v; this allows to work always over the field $\bar{k}$ and not to pass to varieties over the field of Puiseux series.

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2. The universal field

In this section we show existence of the universal valued field for a given valued ring k and a k-algebra R. The proof is based on the following theorem about extension of valuations to a tensor product. If A is a ring, B is an A-algebra, and

v is a valuation of B, then v induces a valuation of A which we call the *restriction* of v and denote $v|_A$.

Theorem 2.1. *Let A be a commutative ring with unity and B and C A-algebras, also with unity. Let u be a valuation of B, v a valuation of C, and assume that $u|_A = v|_A$. Then there exists a valuation w of $B \bigotimes_A C$ that extends both u and v: $w|_B = u$, $w|_C = v$, where B and C map to $B \bigotimes_A C$ as $B \ni b \mapsto b \otimes 1$, $C \ni c \mapsto 1 \otimes c$.*

This is stated without a proof in [Huber, 1.1.14f]. Below we give two proofs[1] of Theorem 2.1. The *first proof* follows from recent stronger results of I. B. Yaacov [Yaacov] and J. Poineau [Poineau]. It is known that in the conditions of Theorem 2.1, the tensor product $B \bigotimes_A C$ carries a natural seminorm $\|\cdot\|$:

$$\|z\| = \inf \max_i (\exp(-u(x_i) - v(y_i))), \quad z = \sum_i x_i \otimes y_i,$$

where the infinum is taken over all representations of $z \in B \bigotimes_A C$ as $\sum x_i \otimes y_i$, $x_i \in B$, $y_i \in C$.

Theorem 2.2 ([Yaacov, Theorem 6])**.** *Let k be a valued algebraically closed field, and K and L two field extensions of k. Assume that K and L are endowed with valuations u and v that restrict to the given valuation of k. Then, the natural seminorm $\|\cdot\|$ of $K \bigotimes_k L$ is multiplicative and the function $-\log\|\cdot\|\colon K \bigotimes_k L \to \overline{\mathbb{R}}$ is a valuation extending both u and v.*

In [Yaacov], this theorem is proven with a help of non-standard technique (ultrapowers) and results on quantifier elimination in some formal theories. It can also be deduced from [Poineau, Section 3], where the technique is the theory of affinoid algebras. To reduce Theorem 2.1 to Theorem 2.2, assume first that the valuation u or v has a nontrivial home. Let $\mathfrak{p} = \mathrm{home}(u) = \{x \in B | u(x) = +\infty\}$, $\mathfrak{q} = \mathrm{home}(v)$. Denote by $\mathfrak{p} \otimes 1$ and $1 \otimes \mathfrak{q}$ the extensions of $\mathfrak{p}$ and $\mathfrak{q}$ to $B \bigotimes_A C$. Then u and v can be considered as valuations on the rings $B/\mathfrak{p}$ and $C/\mathfrak{q}$ respectively, and

$$B/\mathfrak{p} \bigotimes_A C/\mathfrak{q} \simeq (B \bigotimes_A C)/(\mathfrak{p} \otimes 1 + 1 \otimes \mathfrak{q})$$

and $B/\mathfrak{p} \bigotimes_{A/\mathfrak{p}\cap A} C/\mathfrak{q}$ are isomorphic as A-algebras. It follows that we can assume from the beginning that A, B, and C are domains, and $\mathrm{home}(u) = \mathrm{home}(v) = \{0\}$. Thus, u and v extend canonically to the fields of fractions of B and C. Then, passing to the localizations (see, e.g., [AM, Propositions 3.5 and 3.7]), we can assume that $A = k$, $B = K$, and $C = L$ are fields, so it remains only to reduce to an algebraically closed field k. Let $\bar{k}$ be an algebraic closure of k. Since $K' = K \bigotimes_k \bar{k}$ and $L' = L \bigotimes_k \bar{k}$ are integral extensions of K and L respectively, there exist a prime ideal $\mathfrak{p} \subset K'$ lying over $\{0\}$ and a prime ideal $\mathfrak{q} \subset L'$ lying over $\{0\}$. Now, let $\bar{K}$ and $\bar{L}$ be the fields of fractions of $K'/\mathfrak{p}$ and $L'/\mathfrak{q}$ respectively. $\bar{K}$ is an algebraic extension of K, thus, by [Lang, Chapter XII, §3], u extends to $\bar{K}$, and similarly v extends to $\bar{L}$. The tensor product $K \otimes_k L$ maps to $\bar{K} \otimes_{\bar{k}} \bar{L}$, and this map induces embeddings of valued fields $K \subseteq \bar{K}$ and $L \subseteq \bar{L}$. Thus, Theorem 2.1 indeed follows from Theorem 2.2.

The *second proof* of Theorem 2.1 is based on the results of G. M. Bergman. This proof uses a more standard argument of commutative algebra, so we think it is of independent interest. A *pseudovaluation* on a ring R is a map $v\colon R \to \mathbb{R} \cup \{+\infty\}$ satisfying the same axioms as a valuation with the exception that instead of $v(xy) = v(x) + v(y)$ we require only $v(xy) \geq v(x) + v(y)$.

[1]After the submission of this paper, a proof of a more general version (the valuations u and v are not assumed to be of rank 1) has appeared in a preprint *Reified valuations and adic spectra* by K. S. Kedlaya available online on ArXiv.

Theorem 2.3. *Let p be a pseudovaluation on a commutative ring R with unity, and S a multiplicative subsemigroup in $(R,\cdot)$ such that $p|_S$ is a semigroup homomorphism from S to $\overline{\mathbb{R}}$. Let I be an ideal of R such that there is no $s \in S$, $f \in I$ satisfying $v(s) = v(f) < v(f-s)$. Then there exists a valuation $v \geq p$ on R such that $v|_I = +\infty$, $v|_S = p|_S$.*

Proof. This theorem is a direct generalization of [Bergman, Corollary 1]. Indeed, the formula $q(x) = \sup_{f\in I} p(x+f)$ defines a pseudovaluation on R. On the semigroup S, this pseudovaluation coincides with p. Then, by [Bergman, Theorem 2] there exists a valuation $v \geq q$ on R that coincides with q on S. Since $q|_I = +\infty$, we have also $v|_I = +\infty$. □

We continue with an auxiliary lemma.

Lemma 2.4. *Let B and C be two finitely generated A-algebras over a ring A. Let u and v be valuations of B and C respectively that induce the same valuation on A. Finally, let $x_1, \dots, x_m \in B$ and $y_1, \dots, y_n \in C$ be two fixed collections of elements. Then there exists a valuation w on $B \bigotimes_A C$ such that for all $i = 1, \dots, m$ $w(x_i \otimes 1) = u(x_i)$, and for all $j = 1, \dots, n$ $w(1 \otimes y_j) = v(y_j)$.*

Proof. Step 1. First, by the same argument as above we can reduce to the case when $A = k$ is a field, B and C are finitely generated domains over k, and $\mathrm{home}(u) = \{0\}$, $\mathrm{home}(v) = \{0\}$. Next, adding or eliminating some elements if necessary, we can suppose that x_i generate B and y_j generate C as rings over k, and none of x_i, y_j is 0. Represent B as a quotient of a polynomial ring $k[X_1, \dots, X_m]/I$ and C as $k[Y_1, \dots, Y_n]/J$ for ideals $I \subset k[X] = k[X_1, \dots, X_m]$ and $J \subset k[Y] = k[Y_1, \dots, Y_n]$; here $x_i = X_i \mod I$, $y_j = Y_j \mod J$.

Step 2. Let us recall some terminology. We denote also by u the valuation u of B restricted to k ($= v$ of C restricted to k). If $w = (w_1, \dots, w_m) \in \mathbb{R}^n$ is a vector and $f = \sum a_M X^M \in k[X]$ is a polynomial in m variables, the number

$$w(f) = \min_M (u(a_M) + \langle w, M \rangle),$$

where $\langle w, M \rangle = \sum w_i M_i$, is called *$w$-order* of f. The polynomial

$$\mathrm{in}_w(f) = \sum_{u(a_M)+\langle w,M\rangle = w(f)} a_M X^M$$

is called the *w-initial form* of f. For an ideal $I \subset k[X]$, the ideal $\mathrm{in}_w I = (\mathrm{in}_w(f) | f \in I)$ generated by all w-initial forms of elements of I is called the *w-initial ideal* of I.

Now let $w^1 = (u(x_1), \dots, u(x_m)) \in \mathbb{R}^m$, $w^2 = (v(y_1), \dots, w(y_n)) \in \mathbb{R}^n$. Note that since u and v take finite values on x_i and y_j, the initial ideals $\mathrm{in}_{w^1} I$ and $\mathrm{in}_{w^2} J$ are monomial free. Denote $k[X,Y] = k[X_1, \dots, X_m, Y_1, \dots, Y_n]$, $w = (w^1, w^2) \in \mathbb{R}^{m+n}$, and let $I \otimes 1$ and $1 \otimes J$ be the extensions of the ideals I and J to $k[X,Y]$ respectively. Then, $\mathrm{in}_w(I \otimes 1) = \mathrm{in}_{w^1}(I) \otimes 1$, and similarly for J. Indeed, consider $f = \sum f_i b_i \in I \otimes 1$, where $f_i \in k[X,Y]$, $b_i \in I$. Write f as a polynomial in Y: $f = \sum_N b_N Y^N$, where $b_N = b_N(X) \in I$, $b_N \neq 0$. Consider the combination

$$\sum_{w^1(b_N)+\langle w^2,N\rangle \text{ is minimal}} \mathrm{in}_{w^1}(b_N) Y^N. \tag{1}$$

If it is 0, then this holds identically in Y, and thus $\mathrm{in}_{w^1}(b_N) = 0$ for all b_N involved in (1). But this contradicts the assumption $b_N \neq 0$. It follows that the w-initial form of f is a $k[X,Y]$-linear combination of initial forms of $b_N \in I$.

Step 3. Our next claim is that the initial ideal $\mathrm{in}_w(I \otimes 1 + 1 \otimes J)$ coincides with $\mathrm{in}_{w^1}(I) \otimes 1 + 1 \otimes \mathrm{in}_{w^2}(J)$. The inclusion $\supseteq$ is clear, so let us prove the inclusion $\subseteq$.

Consider an element

$$h = \sum_i f_i b_i + \sum_j g_j c_j \in I \otimes 1 + 1 \otimes J,$$

where $f_i, g_j \in k[X,Y]$, $b_i \in I$, $c_j \in J$. We can assume that b_i generate the ideal I, c_j generate the ideal J, the initial forms $\operatorname{in}_{w^1}(b_i)$ generate the initial ideal $\operatorname{in}_{w^1} I$, and $\operatorname{in}_{w^2}(c_j)$ generate $\operatorname{in}_{w^2} J$. If the initial forms do not cancel, that is the combination

$$\sum \operatorname{in}_w(f_i) \operatorname{in}_{w^1}(b_i) + \sum \operatorname{in}_w(g_j) \operatorname{in}_{w^2}(c_j)$$

is not 0, then it is the w-initial form h_w of h, and there is nothing to prove. If this combination is 0, it follows that

$$\sum \operatorname{in}_w(f_i) \operatorname{in}_{w^1}(b_i),\ \sum \operatorname{in}_w(g_j) \operatorname{in}_{w^2}(c_j) \in \operatorname{in}_w(I \otimes 1) \cap \operatorname{in}_w(1 \otimes J),$$

the last intersection being equal to the product $\operatorname{in}_w(I \otimes 1) \cdot \operatorname{in}_w(1 \otimes J)$ by Corollary 2.6 below. Thus, using Step 2, $\sum \operatorname{in}_w(f_i) \operatorname{in}_{w^1}(b_i) = -\sum \operatorname{in}_w(g_j) \operatorname{in}_{w^2}(c_j)$ can be written as $\sum h_{ij} \operatorname{in}_{w_1}(b_i) \operatorname{in}_{w^2}(c_j)$, $h_{ij} \in k[X,Y]$. Then, consider

$$h = \sum f_i b_i + \sum g_j c_j \pm \sum h_{ij} b_i c_j = \sum_i (f_i - \sum_j h_{ij} c_j) b_i + \sum_j (g_j + \sum_i h_{ij} b_i) c_j.$$

The w-order of h is fixed, while w-orders of expressions in parenthesis on the right have increased comparing with the w-orders of f_i and g_j. In this way we eventually rewrite h in the form where w-initial forms do not cancel, thus, w-initial form of h is a combination of w-initial forms of elements of I and J.

Step 4. Now we show that the ideal $\operatorname{in}_w(I \otimes 1 + 1 \otimes J)$ is monomial free. Suppose that a monomial $X^M Y^N$ can be represented as

$$X^M Y^N = \sum f_i \operatorname{in}_{w^1}(b_i) + \sum g_j \operatorname{in}_{w^2}(c_j), \tag{2}$$

where f_i, $g_j \in k[X,Y]$, $b_i \in I$, $c_j \in J$. Since $\operatorname{in}_{w^1} I$ and $\operatorname{in}_{w^2} J$ are monomial free, they have points $x^0 \in (\bar{k}^*)^m$, $y^0 \in (\bar{k}^*)^n$ respectively. Substituting the point $(x^0, y^0) \in (\bar{k}^*)^{m+n}$ to (2), we get a contradiction.

Step 5. Finally, we construct a valuation w on $B \bigotimes_k C$. First, consider a monomial valuation w' on $k[X,Y]$ defined by $w'(X_i) = u(x_i)$, $i = 1, \dots, m$, $w'(Y_j) = v(y_j)$, $j = 1, \dots, n$. Let S be a semigroup generated by all the monomials $aX^M Y^N$, $a \in k$, $M \in \mathbb{Z}^m_{\geq 0}$, $N \in \mathbb{Z}^n_{\geq 0}$. By Step 4 and Bergman's Theorem 2.3, w' can be pushed forward to a valuation w on

$$k[X,Y]/(I \otimes 1 + 1 \otimes J) \simeq k[X]/I \bigotimes_k k[Y]/J \simeq B \bigotimes_k C,$$

and w has all the required properties. □

The following two results were needed for the proof of Lemma 2.4. We use the notation introduced in Step 2 of the proof of Lemma 2.4 and assume some familiarity of the reader with Gröbner bases.

Lemma 2.5. *Let $I \subset k[X]$ be an ideal, and $\{b_1, \dots, b_s\}$ a Gröbner basis of I with respect to some monomial ordering $\leq$ on $k[X]$. Then $\{b_1, \dots, b_s\}$ is a Gröbner basis for $I \otimes 1 \subset k[X,Y]$ with respect to any monomial ordering on $k[X,Y]$ that restricts to the ordering $\leq$ on $k[X]$.*

Proof. For any $f = \sum f_i(X,Y) b_i(X)$ write $f = \sum_N (\sum a_j(X) b_j(X)) Y^N$ as a polynomial in Y. The leading monomial of f is present only in one of the expressions $(\sum a_j b_j) Y^N$, thus it is divisible by the leading monomial of one of the b_j. □

Corollary 2.6. *Let $I \subset k[X]$ and $J \subset k[Y]$ be ideals. Then*

$$(I \otimes 1) \cap (1 \otimes J) = (I \otimes 1) \cdot (1 \otimes J)$$

in $k[X, Y]$.

Proof. Fix a monomial ordering on $k[X, Y]$ and Gröbner bases $\{b_i\}$ for I and $\{c_j\}$ for J. Take $f \in (I \otimes 1) \cap (1 \otimes J)$. By Lemma 2.5, we conclude that the leading monomial aX^MY^N, $a \in k$, of f is divisible by the leading monomial of one of b_i and of one of c_j. But the leading monomial of each b_i is coprime to the leading monomial of each c_j, thus aX^MY^N is divisible by the leading monomial of some of the products $b_i(X)c_j(Y)$. This implies the corollary. □

Let us continue the proof of Theorem 2.1. Now we consider the general case of the tensor product of A-algebras B and C, not necessarily finitely generated over A. It is again possible to reduce to the case when $A = k$ is a field, and this will be assumed in the sequel. Let us recall the construction of the tensor product. Denote by $B^\bullet$ and $C^\bullet$ the multiplicative semigroups of the rings B and C respectively, and consider the direct product of semigroups $S' = B^\bullet \times C^\bullet$. Then the tensor product $B \bigotimes_k C$ can be identified with the quotient $k[S']/T$ of the semigroup algebra $k[S']$ by the ideal T generated by all relations of the form

$$(ax, y) - a(x, y),\ (x, ay) - a(x, y),$$
$$(x' + x'', y) - (x', y) - (x'', y),\ (x, y' + y'') - (x, y') - (x, y''), \tag{3}$$

where $a \in k$, $x, x', x'' \in B$, $y, y', y'' \in C$. Note that these expressions generate J not only as an ideal of $k[S']$ but also as a vector space over k.

The valuations u and v of B and C are semigroup homomorphisms $u\colon B^\bullet \to \overline{\mathbb{R}}$, $v\colon C^\bullet \to \overline{\mathbb{R}}$. The rule $p(x, y) = u(x) + v(x)$ defines a semigroup homomorphism $p\colon S' \to \overline{\mathbb{R}}$. Furthermore, we extend p to a pseudovaluation on the semigroup algebra $k[S']$. For $f = \sum a_s s \in k[S']$, where $a_s \in k$, $s \in S'$, we set

$$p(f) = \min_{s \in S'} (u(a_s) + p(s))$$

(we could write $v(a_s)$ instead of $u(a_s)$). Let S be a subsemigroup of $k[S']$ generated by all the monomials $a(x, y)$, $a \in k$, $a \neq 0$, $(x, y) \in S$. It is clear that $p|_S$ is a semigroup homomorphism. Next we are going to show that there exists a valuation $\bar{w}$ on $k[S']$ that coincides with p on S and can be pushed forward to a valuation w of the quotient $k[S']/T$. It suffices only to check the conditions of Theorem 2.3, i.e., if $f \in T$ and $s \in S$ is a monomial, then $p(f - s) \leq p(f)$. Represent f as a k-linear combination $\sum a_i r_i$ of expressions r_i of the form (3), and s as $a(x_0, y_0)$, where $a \in k$, $x_0 \in B$, $y_0 \in C$. Let $x_1, \dots, x_m$ be all the elements of B and $y_1, \dots, y_n$ all the elements of C that are present in the monomials of r_i. Consider the finitely generated rings $B' = k[x_0, x_1, \dots, x_m]$ and $C' = k[y_0, y_1, \dots, y_n]$ and restrictions u' and v' to B' and C' respectively. If we represent $B' \bigotimes_k C'$ as a quotient of a semigroup algebra $k[(B')^\bullet \times (C')^\bullet]$, then f can be considered as an element of the corresponding ideal $T' \subset k[(B')^\bullet \times (C')^\bullet]$. The pseudovaluation p also naturally restricts to this semigroup algebra. On the other hand, we know by Lemma 2.4 that u' and v' extend simultaneously to $B' \bigotimes_k C'$, thus, p achieves its minimum on the monomials of f at least twice. This shows that Theorem 2.3 applies to p, S and T. We get a valuation w on

$$k[S']/T \simeq B \bigotimes_k C$$

which is a simultaneous extension of u and v. This finishes the second proof of Theorem 2.1.

We are ready to describe the construction of universal valued fields. Now, let k be a ring with a real valuation ν and R an arbitrary commutative k-algebra with unity. Denote by $\mathcal{V}(R,k)$ the set of all real ring valuations of R that extend ν.

Theorem 2.7. *Given k, R, and ν, there exist a k-algebra K and a valuation μ on K such that $\mu|_k = \nu$, and for any $v \in \mathcal{V}(R,k)$ there exists a morphism $f_v \colon R \to K$ of k-algebras such that the valuation v on R is induced by the valuation μ on K via the morphism f_v.*

Proof. Consider a k-algebra

$$K = \bigotimes_{v\in\mathcal{V}(R,k)} R,$$

the restricted tensor product over k of the k-algebra R with itself, one copy for each $v \in \mathcal{V}(R,k)$, see [Eisenbud, p. 713, Proposition A6.7b]. Let us show that there exists a valuation μ on K that is a simultaneous extension of all the valuations v of R. This is a consequence of Theorem 2.1 and Zorn's lemma. Indeed, for each subset $V \subseteq \mathcal{V}(R,k)$ we have a natural homomorphism of k-algebras

$$K_V = \bigotimes_{v\in V} R \to K,$$

and, for $U \subseteq V$, a natural homomorphism $K_U \to K_V$. Let $\mathcal{V}$ be the family of all pairs (V, μ_V), where $V \subseteq \mathcal{V}(R,k)$ and μ_V is a valuation on K_V extending all the valuations v of R, $v \in V$. The family $\mathcal{V}$ is nonempty since it contains all the pairs $(\{v\}, v)$ for one-element subsets of $\mathcal{V}(R,k)$. It is also ordered by the following order relation: $(U,\mu_U) \le (V,\mu_V)$ if and only if $U \subseteq V$ and $(\mu_V)|_{K_U} = \mu_U$. Then, the conditions of Zorn's lemma are satisfied, thus $\mathcal{V}$ contains maximal elements. But by Theorem 2.1 such a maximal element must coincide with $(\mathcal{V}(R,k),\mu)$ for some valuation μ on K. Now, let f_v be the natural homomorphism

$$R \to \bigotimes_{v\in\mathcal{V}(R,k)} R = K.$$

Then, μ induces the valuation v on R via f_v. □

Remark 2.8. In the conditions of Theorem 2.7, the valuation μ on K naturally extends to $K/\operatorname{home}(\mu)$, to its field of fractions $Q(K/\operatorname{home}(\mu))$, and, further, to its algebraic closure $\overline{Q(K/\operatorname{home}(\mu))}$. Thus, the universal k-algebra K can be assumed to be a domain or even an algebraically closed field.

It would be interesting to give an explicit construction of universal valuation fields under some reasonable restrictions on the algebra R. It may be that such a universal valued field depends only on the field k and the valuation ν and works for all k-algebras of a certain class. The next proposition gives an example of such phenomenon.

Proposition 2.9. *Let k be an algebraically closed trivially valued field of characteristic 0, and $k(x) = k(x_1,\dots,x_n)$ the field of rational functions in n variables with coefficients in k. Let $K = \overline{k(x)}(t^{\mathbb{R}})$ be the Hahn series field with coefficients in the algebraic closure of $k(x)$. Then K serves as a universal valued field for all finitely generated k-algebras R of dimension $d \le n$.*

Proof. Let R be a finitely generated k-algebra of dimension $d \le n$. If v is a valuation of R over k, then the residue field $k(v)$ of v has transcendence degree $r \le d$ over k. Thus $k(v)$ and its algebraic closure can be embedded to $\overline{k(x)}$. On the other hand, by [Kaplansky, Theorem 6], the field $\overline{k(v)}(t^{\mathbb{R}})$ is the maximal valued field with the residue field $\overline{k(v)}$ and the value group $\mathbb{R}$. It follows that the valuation v is induced by a homomorphism from R to $\overline{k(v)}(t^{\mathbb{R}})$. It remains to note that each field of the form $\overline{k(v)}(t^{\mathbb{R}})$ embeds to K. □

Proposition 2.9 suffers obviously from the lack of explicit description of the algebraic closure $\overline{k(x)}$ of the field of rational functions in several variables.

3. Weak universality of the field of Puiseux series

As in the Introduction, we denote by $K = \bar{k}((t^{\mathbb{Q}})) = \bigcup_{N\geq 1} \bar{k}((t^{1/N}))$ the field of Puiseux series with coefficients in an algebraically closed field $\bar{k}$, and by $K' = \bar{k}((t^{\mathbb{R}}))$ the field of Hahn series, i.e., the set of all formal sums

$$a(t) = \sum_{m\in\mathbb{R}} a_m t^m,$$

where $a_m \in \bar{k}$ and $\{m \,|\, a_m \neq 0\}$ is a well-ordered subset of $\mathbb{R}$. Let μ be the natural valuation of K (K'). The ring of Puiseux (respectively Hahn) series $\bar{k}[[t^{\mathbb{Q}}]]$ (resp. $\bar{k}[[t^{\mathbb{R}}]]$) is the subring of K (resp. K') consisting of the series of nonnegative valuation. It is known that if $\bar{k}$ has characteristic 0, then the field K of Puiseux series is algebraically closed ([Cohn]); if $\bar{k} = \mathbb{C}$ is the field of complex numbers, this is the classical Newton-Puiseux Theorem. The field K' of Hahn series is always algebraically closed ([MacLane, Theorem 1]).

Theorem 3.1. *Let R be a complete Noetherian local ring containing a field k isomorphic to the residue field of R. Let v be a local valuation of R and $x_1, \dots, x_n$ a finite collection of elements of the maximal ideal of R. Then there exists a homomorphism $f\colon R \to \mathcal{O}' = \bar{k}[[t^{\mathbb{R}}]]$ to the ring of Hahn series with coefficients in $\bar{k}$ such that for all $i = 1, \dots, n$, $v(x_i) = \mu(f(x_i))$. Moreover, if the characteristic of k is 0, the elements $x_1, \dots, x_n$ analytically generate R, and $v(x_i) \in \mathbb{Q}$ for all $i = 1, \dots, n$, then there exists a homomorphism $f\colon R \to \mathcal{O} = \bar{k}[[t^{\mathbb{Q}}]]$ to the ring of Puiseux series with coefficients in $\bar{k}$ with the same property.*

Proof. Step 1. First we reduce to the case when $k = \bar{k}$ is algebraically closed, R is a domain, and do some other preliminary reductions. Consider the tensor product $\overline{R} = R \bigotimes_k \bar{k}$. Note that $\overline{R}$ is integral over R, thus, there exists a prime ideal $\mathfrak{q} \subset \overline{R}$ such that $\mathfrak{q} \cap R = \mathfrak{p}$, where $\mathfrak{p} = \mathrm{home}(v)$. The quotient $\overline{R}/\mathfrak{q}$ remains integral over $R/\mathfrak{p}$. Hence the local valuation v of $R/\mathfrak{p}$ extends to a valuation $\bar{v}$ of $\overline{R}/\mathfrak{q}$, and, as follows from the method of Newton polygon (see [Bourbaki, Chapitre VI, §4, exercise 11]), $\bar{v}$ is nonnegative on $\overline{R}/\mathfrak{q}$. Passing again to the quotient by $\mathrm{home}(\bar{v})$, if necessary, we can assume that $\mathrm{home}(\bar{v}) = \{0\}$. Let $\mathfrak{n}$ be the prime ideal of $\overline{R}/\mathfrak{q}$ where $\bar{v}$ is positive. Since $\mathfrak{n} \cap R/\mathfrak{p} = \mathfrak{m}$ is the maximal ideal of $R/\mathfrak{p}$, the ideal $\mathfrak{n}$ is also maximal. Then, consider the localization $(\overline{R}/\mathfrak{q})_{\mathfrak{n}}$ and its completion R' with respect to $\mathfrak{n}$. The valuation $\bar{v}$ extends canonically to a local valuation v' of R' (see, e.g., [PPS, Lemma 5.16]). Finally, passing to the quotient of R' by $\mathrm{home}(v')$, we can assume that R' is a domain and $\mathrm{home}(v') = \{0\}$.

Recall that the *rational rank* of a valuation v on R is the dimension over $\mathbb{Q}$ of the $\mathbb{Q}$-vector subspace of $\mathbb{R}$ generated by the image of v. Let r' be the rational rank of v and d the Krull dimension of R. It follows from the Abhyankar inequality ([Vaquié, Théorème 9.2]) that $r' \leq d$. If r is the dimension of the $\mathbb{Q}$-vector subspace of $\mathbb{R}$ generated by $v(x_1), \dots, v(x_n)$, then we have $r \leq r' \leq d$. By the Cohen structure theorem [Eisenbud, Theorem 7.7], the ring R is a quotient of the ring $k[[X_1, \dots, X_N]]$ of formal power series. Enlarging the finite set $\{x_1, \dots, x_n\}$, if necessary, we can assume that the elements $x_1, \dots, x_n$ analytically generate R, i.e., $N = n$ and a surjection from $k[[X_1, \dots, X_n]]$ to R can be chosen so that $X_i \mapsto x_i$, $i = 1, \dots, n$. Clearly, there is no loss of generality if we also assume that none of x_i is 0.

In the reduction steps described above, we performed the following actions with the ring R: we took quotients, considered the tensor product with $\bar{k}$, localized at a maximal ideal, and took a completion. These actions could only decrease the dimension of R. Thus, if the elements $x_1, \dots, x_n$ analytically generate the ring R at the beginning, their images in R' form a system of parameters for R'. Then, a system of analytical generators for R' can be obtained from $x_1, \dots, x_n$ by adjoining only a finite number of elements $x'_1, \dots, x'_l \in R'$ *integral* over the subring analytically generated by $x_1, \dots, x_n$. It again follows from the method of Newton polygon that if v takes only rational values on x_i, then it also takes only rational values on x'_j, $j = 1, \dots, l$. This shows that the reduction steps are applicable also in the "moreover" part of the theorem, i.e., we can assume that R is a local complete Noetherian domain, k is an algebraically closed subfield of R of characteristic 0 and isomorphic to the residue field of R, $x_1, \dots, x_n$ analytically generate R, and v takes only rational values on x_i.

Step 2. Now, we reduce to the case $r = d$, where r is the dimension of the $\mathbb{Q}$-vector subspace of $\mathbb{R}$ generated by $v(x_1), \dots, v(x_n)$, $d = \dim R$. Suppose that $r < d$. Then, represent R as a quotient $k[[X_1, \dots, X_n]]/I$, where I is a prime ideal of $k[[X_1, \dots, X_n]]$. The valuation v induces a valuation of $k[[X_1, \dots, X_n]]$; by abuse of notation, we denote this induced valuation also by v. Let w be the vector $(w_1, \dots, w_n) = (v(x_1), \dots, v(x_n)) \in \mathbb{R}^n$. The proof of the following elementary lemma is left to the reader.

Lemma 3.2. *Let $w = (w_1, \dots, w_n) \in \mathbb{R}^n$ be a vector such that the real numbers $w_1, \dots, w_n$ generate a $\mathbb{Q}$-vector subspace of $\mathbb{R}$ of dimension r over $\mathbb{Q}$. Then there exists a unique minimal rational vector subspace $L_{\mathbb{Q}}(w)$ of $\mathbb{R}^n$ (*rational *means that $L_{\mathbb{Q}}(w)$ is defined by linear equations with rational coefficients) such that $w \in L_{\mathbb{Q}}(w)$. The dimension of $L_{\mathbb{Q}}(w)$ equals r.*

By general properties of valuations, the initial ideal $\operatorname{in}_w I$ is monomial free. Consider the local Gröbner fan of the ideal I ([BT]). There exists a rational polyhedral cone $\sigma \subset \mathbb{R}^n$ such that w is contained in the relative interior $\mathring{\sigma}$ of σ and for all $w' \in \mathring{\sigma}$, $\operatorname{in}_{w'} I = \operatorname{in}_w I$. The intersection $\sigma \cap L_{\mathbb{Q}}(w)$ is also a rational polyhedral cone. Let us fix a vector $w' \in \mathring{\sigma} \cap L_{\mathbb{Q}}(w)$ with positive *integral* coordinates. Note that the initial ideal $\operatorname{in}_{w'} I = \operatorname{in}_w I$ is generated by w-homogeneous polynomials. Then, let $J = \operatorname{in}_{w'} I \cap k[x_1, \dots, x_n]$ be the corresponding ideal of the polynomial ring. We have

$$k[[x_1, \dots, x_n]]/\operatorname{in}_{w'} I \simeq (k[x_1, \dots, x_n]/J)\widehat{\ },$$

where the completion is taken with respect to the maximal ideal $(x_1, \dots, x_n)$. Thus, by [AM, Corollary 11.19],

$$\dim k[[x_1, \dots, x_n]]/\operatorname{in}_{w'} I = \dim k[x_1, \dots, x_n]/J.$$

But by the standard flat degeneration argument (see, e.g., [PPS, Lemma 11.10]), $\dim k[[x_1, \dots, x_n]]/\operatorname{in}_{w'} I = \dim R$, and, since I is supposed to be prime, all minimal associated primes of $\operatorname{in}_{w'} I$ have the same depth $d = \dim R$. We conclude that $\dim k[x_1, \dots, x_n]/J = d$, and, moreover, all minimal associated primes of J also have depth d.

Since the ideal J is monomial free, it has a point $x^0 \in (k^*)^n$. Choose an irreducible component Z of the zero set of J such that $x^0 \in Z$. Let $\gamma_1, \dots, \gamma_r$ be a basis of the $\mathbb{Q}$-vector subspace of $\mathbb{R}$ generated by $w_1, \dots, w_n$. Write

$$w_i = \sum_{j=1}^{r} w_{ij} \gamma_j,$$

where $i = 1, \dots, n$, $w_{ij} \in \mathbb{Q}$. Rescaling γ_j if necessary, we can even suppose that w_{ij} are integral. By our assumptions, the rank of the matrix $(w_{ij})_{i=1,j=1}^{n,m}$ is r. Since the ideal J is generated by w-homogeneous polynomials, together with the point $x^0 = (x_1^0, \dots, x_n^0)$ the variety Z contains also all the points

$$(t_1^{w_{11}} \cdots t_r^{w_{1r}} x_1^0, \dots, t_1^{w_{n1}} \cdots t_r^{w_{nr}} x_n^0), \quad (t_1, \dots, t_r) \in k^r.$$

Thus we can make some r of the coordinates of x^0, say, $x_1^0, \dots, x_r^0$, to be equal to 1. The dimension of the linear affine space H: $x_1 = \cdots = x_r = 1$ is $n - r$, and the dimension of Z is $d > r$. Hence $\dim H \cap Z \geq 1$, and there exists at least one point $y^0 \neq x^0$, $y^0 \in H \cap Z$. Take a polynomial $\tilde{f}(x_{r+1}, \dots, x_n)$ such that $\tilde{f}(x_{r+1}^0, \dots, x_n^0) = 0$, $\tilde{f}(y_{r+1}^0, \dots, x_n^0) \neq 0$. Homogenizing $\tilde{f}$ with respect to the weight vector w' we get a w'-homogeneous polynomial f such that $f(x^0) = 0$, $f(y^0) \neq 0$. By construction, f is also w-homogeneous, not a monomial, and not contained in any of the minimal associated primes of $\operatorname{in}_w I$.

Lemma 3.3. *Let I, w, and f be as above. Then, in the ring $k[[x_1, \dots, x_n]]$,*

$$\operatorname{in}_w(I + (f)) = \operatorname{in}_w I + (f).$$

Proof. Consider an arbitrary $fg + h \in (f) + I$, $h \in I$, $g \in k[[x_1, \dots, x_n]]$. If $f \cdot \operatorname{in}_w g + \operatorname{in}_w h \neq 0$, then it is the w-initial form of $fg + h$ and is contained in $\operatorname{in}_w I + (f)$. Assume that $f \cdot \operatorname{in}_w g + \operatorname{in}_w h = 0$. Since f is not a zero divisor modulo $\operatorname{in}_w I$, we have $\operatorname{in}_w g \in \operatorname{in}_w I$. Let $\tilde{g} \in I$ be such that $\operatorname{in}_w \tilde{g} = \operatorname{in}_w g$. Then $fg + h$ has the same initial form as

$$fg \pm f\tilde{g} + h = fg' + h',$$

where $h' \in I$ and g' has w-order strictly greater than g. Repeating this argument we come to the situation when the initial forms of fg' and h' do not cancel, and thus $\operatorname{in}_w(fg + h) \in \operatorname{in}_w I + (f)$. This proves the inclusion

$$\operatorname{in}_w(I + (f)) \subseteq \operatorname{in}_w I + (f).$$

The inverse inclusion is clear. □

By construction, the ideal $\operatorname{in}_w(I + (f)) = \operatorname{in}_w I + (f)$ is monomial free. By Bergman's Theorem 2.3, the monomial valuation v' on $k[[x_1, \dots, x_n]]$ defined by $v'(x_i) = w_i$, $i = 1, \dots, n$, can be transformed to a valuation $\bar{v}$ such that $\bar{v}(x_i) = w_i$, $i = 1, \dots, n$, and $\bar{v}|_{I+(f)} = +\infty$. Therefore we can pass to the ring $R' = R/(f)$, $\dim R' = d - 1$.

Step 3. It remains to consider the case $d = \dim R = r$. The rational rank r' of the valuation v is not greater than d by Abhyankar inequality, so in this case $r = r'$. Choose a system of parameters $y_1, \dots, y_d$ for R such that $v_1 = v(y_1)$, …, $v_d = v(y_d)$ are linearly independent over $\mathbb{Q}$. The ring R is integral over $k[[y_1, \dots, y_d]]$.

Embed $k[[y_1, \dots, y_d]]$ to the ring of Hahn series $\mathcal{O}' \subset k((t^{\mathbb{R}}))$ by sending y_i to t^{v_i}, $i = 1, \dots, d$. The natural valuation μ of $k((t^{\mathbb{R}}))$ is an extension of v from $k[[y_1, \dots, y_d]]$ via this embedding. By [Lang, Chapter XII, §3], every extension of v from $k[[y_1, \dots, y_d]]$ to R can be induced by an embedding of R to

$$\overline{Q(k[[y_1, \dots, y_d]])_v},$$

i.e., to the algebraic closure of the completion with respect to v of the field of fractions of $k[[y_1, \dots, y_d]]$. Since $k((t^{\mathbb{R}}))$ is algebraically closed and complete, we conclude that there exists an embedding of R to $k((t^{\mathbb{R}}))$ such that μ induces v. Moreover, the method of Newton polygon shows that the image of R is contained in $\mathcal{O}'$.

Finally, assume that v takes only rational values on x_i and the field k has characteristic 0. This implies, in particular, that $r = d = 1$. This time we embed $k[[y]] = k[[y_1]]$ to the ring of Puiseux series $\mathcal{O}$ by sending y to t^{v_1}. As above, we get an embedding of R to $k((t^{\mathbb{R}}))$ inducing the valuation v, but, since the field $k((t^{\mathbb{Q}}))$

is also algebraically closed, the image is contained in $k((t^{\mathbb{Q}}))$ and, by the method of Newton polygon, in $\mathcal{O}$. □

The following theorem shows weak universality of the field of Hahn series with respect to valuations on finitely generated algebras.

Theorem 3.4. *Let R be a finitely generated algebra over a field k. Let v be a valuation of R and $x_1, \ldots, x_n$ a finite collection of elements of R. Then there exists a homomorphism $f\colon R \to \bar{k}((t^{\mathbb{R}}))$ to the field of Hahn series with coefficients in $\bar{k}$ such that for all $i = 1, \ldots, n$, $v(x_i) = \mu(f(x_i))$. Moreover, if the characteristic of k is 0, the elements $x_1, \ldots, x_n$ generate R over k, and $v(x_i) \in \mathbb{Q}$ for all $i = 1, \ldots, n$, then there exists a homomorphism $f\colon R \to \bar{k}((t^{\mathbb{Q}}))$ to the field of Puiseux series with coefficients in $\bar{k}$ with the same property.*

Proof. This theorem follows from the results on lifting points in tropical varieties, see, e.g., [JMM]. Alternatively, it can be proven by an argument parallel to the proof of Theorem 3.1. □

4. Lifting points in local tropical varieties

In this section we recall briefly the definition of local tropicalization following [PPS] and deduce the lifting points property of local tropical varieties from Theorem 3.1. Let M be a finitely generated free abelian group and $M_{\mathbb{R}}$ the vector space $M \bigotimes_Z \mathbb{R}$. For rational polyhedral cones $\check{\sigma}$ in $M_{\mathbb{R}}$, we consider additive semigroups Γ of the form $\check{\sigma} \cap M$ or, more generally, finitely generated subsemigroups (with neutral element) of $\check{\sigma} \cap M$ such that $\mathrm{Sat}(\Gamma)$, the saturation of Γ (see [PPS, Definition 2.10]), equals $\check{\sigma} \cap M$. Each semigroup homomorphism from Γ to $\mathbb{R}$ lifts uniquely to a group homomorphism from M to $\mathbb{R}$. Thus we can identify the group $\mathrm{Hom}_{sg}(\Gamma, \mathbb{R})$ of all semigroup homomorphisms from Γ to $\mathbb{R}$ with the dual space of $M_{\mathbb{R}}$; we denote this dual space by $N_{\mathbb{R}}$, and by N the dual lattice of M. The semigroup homomorphisms from Γ to the extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ live in a certain partial compactification of $N_{\mathbb{R}}$ that is denoted by $L(\sigma, N)$ in [PPS, Section 4] and called the *linear variety* corresponding to σ and N; here σ is the dual cone of $\check{\sigma}$. The cone σ corresponds to nonnegative homomorphisms from Γ to $\mathbb{R}$; the nonnegative homomorphisms from Γ to $\overline{\mathbb{R}}$ form a certain subspace of $L(\sigma, N)$ denoted $\overline{\sigma}$. We denote by $\overline{\sigma}^{\circ}$ the interior of $\overline{\sigma}^{\circ}$.

Now, let R be a local ring with the maximal ideal $\mathfrak{m}$. Let $\mathcal{V}_{\geq 0}(R)$ be the set of all nonnegative real valuations of R and $\mathcal{V}_{>0}(R)$ the set of all local valuations of R, i.e., valuations that are nonnegative on R and positive on the maximal ideal $\mathfrak{m}$. Let $\gamma\colon \Gamma \to R$ be a homomorphism from Γ to the multiplicative semigroup of R and assume that none of the elements of $\gamma^{-1}(\mathfrak{m})$ is invertible in Γ. Then, for each $v \in \mathcal{V}_{\geq 0}$ (resp. $v \in \mathcal{V}_{>0}$), the composition $v \circ \gamma$ is an element of $\overline{\sigma}$ (resp. $\overline{\sigma}^{\circ}$). In this way we get a map

$$\mathrm{Trop}\colon \mathcal{V}_{\geq 0}(R) \to \overline{\sigma} \quad (\text{resp. } \mathrm{Trop}\colon \mathcal{V}_{>0}(R) \to \overline{\sigma}^{\circ}),$$

which we call the *tropicalization map.*

Definition 4.1. The *local nonnegative tropicalization* of the morphism γ, denoted $\mathrm{Trop}_{\geq 0}(\gamma)$, is the image of $\mathcal{V}_{\geq 0}(R)$ in $\overline{\sigma}$ under the tropicalization map Trop. The *local positive tropicalization*, denoted $\mathrm{Trop}_{>0}(\gamma)$, is the closure in $\overline{\sigma}^{\circ}$ of the image of $\mathcal{V}_{>0}(R)$ under the tropicalization map.

Next we restrict to local rings R of special type. Suppose that the cone $\check{\sigma}$ is *strictly convex*, in particular, the only invertible element of Γ is 0. Let k be a field and consider all possible formal series

$$\sum_{m\in\Gamma} a_m \chi^m,$$

where $a_m \in k$. Any two such series can be added and multiplied in a natural way, in particular, $\chi^m \cdot \chi^{m'} = \chi^{m+m'}$. These series form a complete Noetherian local ring $k[[\Gamma]]$ called the *ring of formal power series* over Γ, see [PPS, Section 8]. There is a natural semigroup homomorphism $\Gamma \to k[[\Gamma]]$ defined by $\Gamma \ni m \mapsto \chi^m$, the elements $\chi^m \in k[[\Gamma]]$ being called the *monomials*. If $I \subset k[[\Gamma]]$ is an ideal, let γ be the composition of the natural maps

$$\Gamma \to k[[\Gamma]] \to R = k[[\Gamma]]/I.$$

Now, we can consider the nonnegative (resp. positive) tropicalization $\mathrm{Trop}_{\geq 0}(\gamma)$ (resp. $\mathrm{Trop}_{>0}(\gamma)$) of γ. In the situation just described, we denote also $\mathrm{Trop}_{\geq 0}(\gamma) = \mathrm{Trop}_{\geq 0}(I)$ ($\mathrm{Trop}_{>0}(\gamma) = \mathrm{Trop}_{>0}(I)$) and call it the *local nonnegative* (resp. *positive*) *tropicalization of the ideal I*.

It is proven in [PPS, Theorem 11.9] that both nonnegative $\mathrm{Trop}_{\geq 0}(I) \subseteq \overline{\sigma}$ and positive $\mathrm{Trop}_{>0}(I) \subseteq \overline{\sigma}^\circ$ tropicalizations are rational PL conical subspaces of real dimension d equal to the Krull dimension of R. This means, in particular, that the part of $\mathrm{Trop}_{\geq 0}(I)$ contained in $N_\mathbb{R}$ is a support of a rational (with respect to the lattice N) polyhedral fan. The nonnegative tropicalization $\mathrm{Trop}_{\geq 0}(I)$ is stratified by the positive tropicalizations $\mathrm{Trop}_{>0}(I_\tau)$ of certain truncations of the ring R, see [PPS, Section 12, in particular Lemma 12.9] for the details.

Theorem 4.2 (Lifting points lemma for local tropical varieties)**.** *Let $I \subset k[[\Gamma]]$ be an ideal. If $u \in \mathrm{Trop}_{>0}(I)$, then there exists a homomorphism $f\colon R = k[[\Gamma]]/I \to \bar{k}[[t^\mathbb{R}]]$ to the ring of Hahn series with coefficients in $\bar{k}$ such that $u = \mathrm{Trop}(\mu \circ f)$, where μ is the natural valuation of $\bar{k}[[t^\mathbb{R}]]$. Moreover, if the field k has characteristic 0 and the point u is rational, then there exists a homomorphism $f\colon R \to \bar{k}[[t^\mathbb{Q}]]$ to the ring of Puiseux series with the same property.*

Proof. By [PPS, Theorem 11.2], the image of $\mathcal{V}_{>0}(R)$ in $\overline{\sigma}^\circ$ is closed under the tropicalization map, hence there exists $v \in \mathcal{V}_{>0}(R)$ such that $u = \mathrm{Trop}(v)$. If $u \in \mathrm{Trop}_{>0}(I) \subseteq \overline{\sigma}^\circ$ belongs to some stratum of $\overline{\sigma}$ at infinity, then v is induced by a valuation on a quotient R/J of the ring R, where R/J also has the form $k[[\Gamma']]$ for some semigroup Γ', see [PPS, the end of Section 8]. Therefore it suffices to consider the case $u \in N_\mathbb{R}$. Choose monomials $x_1, \ldots, x_n \in \Gamma$ which generate the semigroup Γ. A semigroup homomorphism from Γ to $\mathbb{R}$ is uniquely determined by the images of $x_1, \ldots, x_n$. By Theorem 3.1 there exists a homomorphism $f\colon R \to \bar{k}[[t^\mathbb{R}]]$ such that $\mu(f(x_i)) = u(x_i)$ for all $i = 1, \ldots, n$. If the characteristic of k is 0 and u is rational, a homomorphism f can be chosen so that the target ring is $\bar{k}[[t^\mathbb{Q}]]$. This f is the required homomorphism. □

Corollary 4.3. *Let $I \subset k[[\Gamma]]$ be an ideal. The following three definitions of the local positive tropicalization of I are equivalent:*

1) $\mathrm{Trop}_{>0}(I)$ *is the image of $\mathcal{V}_{>0}(k[[\Gamma]]/I)$ under the tropicalization map;*
2) $\mathrm{Trop}_{>0}(I)$ *is the set of all $w \in \overline{\sigma}^\circ$ such that the initial ideal $\mathrm{in}_w I$ is monomial free;*
3) $\mathrm{Trop}_{>0}(I)$ *is the set $\{\mathrm{Trop}(\mu \circ f) \,|\, f\colon R \to \bar{k}[[t^\mathbb{R}]]\}$, where f runs over all local homomorphisms from $R = k[[\Gamma]]/I$ to $\bar{k}[[t^\mathbb{R}]]$.*

If the field k has characteristic 0, then $\mathrm{Trop}_{>0}(I)$ can also be described as

3′) *the closure of the set $\{\mathrm{Trop}(\mu \circ f) \,|\, f\colon R \to \bar{k}[[t^\mathbb{Q}]]\}$ in $\overline{\sigma}^\circ$, where f runs over all local homomorphisms from R to $\bar{k}[[t^\mathbb{Q}]]$.*

Proof. Equivalence of 1) and 2) is proven in [PPS, Theorem 11.2]. Equivalence of 1) and 3), and, if the characteristic of k is 0, of 1) and 3′), follows from Theorem 4.2 and from the fact that the local positive tropicalization defined by 1) is a *rational* conical PL space, [PPS, Theorem 11.9]. □

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